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1983 J. Phys. A: Math. Gen. 16 L713

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## LETTER TO THE EDITOR

### The thermodynamic limit on Bethe lattices†

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Received 29 September 1983

**Abstract.** We emphasise the fact that, to obtain the exact solution of Hamiltonian models on Bethe lattices, one can apply explicitly the formal method of rigorous statistical mechanics, i.e. the thermodynamic limit of probability measures. This approach solves the well known dichotomy between clashing alternative solutions, and provides a very simple analytic solution for a large class of Hamiltonian models.

Bethe lattices, i.e. infinite connected trees whose sites have the same coordination number (for graph theory terminology we refer to Essam and Fisher (1970)), deserve some interest in statistical mechanics owing to their thin structure. For every  $m$ -step Markov Hamiltonian model (see e.g. Sinai (1982); also see Feller (1968) and Loève (1977)) only one bundle of correlation paths exists around the unique walk between any pair of sites. This means that, in principle, all correlation functions can be calculated exactly, thus solving the model under investigation. Due to this property, certain Hamiltonian models (mostly one-step Markov, as the Ising model with only nearest-neighbour interactions and external fields) have been considered in detail on this kind of lattice (Kurata *et al* 1953, Domb 1960, Woodbury 1967, Runnels 1967, Obokata and Oguchi 1968, Wheeler and Widom 1970, Eggarter 1974, von Heimbürg and Thomas 1974, Matsuda 1974, Katsura and Takizawa 1974, Katsura 1975, Müller-Hartmann and Zittartz 1974, 1975, Müller-Hartmann 1977, Falk 1975, Coniglio 1975, 1976, Muto and Oguchi 1976, Wang and Wu 1976, Moraal 1976, 1978, 1981, 1982a, b, c, d, e, Morita and Horiguchi 1981, Thompson 1982, Baumgärtel and Müller-Hartmann 1982)§. It is known (see e.g. Baxter 1982) that distinct solutions can be obtained for the same model when the thermodynamic limit on increasing sequences of finite trees, whose union is the Bethe lattice, is taken. Roughly speaking, the differences arise according to whether, in the calculation of extensive functions, one looks for (i) the properties of the complete trees, or (ii) only those of their interior. The so-called Cayley tree solutions (i) use the thermodynamic limit of the free energy and take into account the non-negligible surface effects which arise in the process. The so-called Bethe lattice solutions (ii) use the partition function of finite systems (but do not consider at all surface effects), local magnetisations and uniformity requirements, and subsequently take the thermodynamic limit. We emphasise the fact that both approaches (i) and (ii) are formally unsatisfactory. It is well known that, on

† Supported by MPI and CNR.

‡ Gruppo Nazionale di Struttura della Materia.

§ This set of references is not exhaustive.

$d$ -dimensional hypercubic lattices and in other systems, the correct free energy limits are obtained when van Hove's convergence condition holds (Ruelle 1969, 1978, Huang 1963). The fact that no increasing sequence of trees tends to the Bethe lattice in the sense of van Hove (see below) does not imply that we are bound to accept the results of type (i) methods, nor that they are exact. On the other hand, type (ii) methods do not give formal justifications for the deletion of surface effects and sometimes reduce to applications of the Bethe–Peierls cluster approximation (Bethe 1935, Peierls 1936), although no rigorous proof was given yet that it is exact on Bethe lattices, except for some consistency checks given in special cases (Kurata *et al* 1953, Domb 1960, Katsura and Takizawa 1974, Wang and Wu 1976). However, it is known that the canonical approach of the thermodynamic formalism (Ruelle 1978), i.e. the thermodynamic limit of probability measures (PM), is totally unaffected by the properties of the sequences used and the lattice considered. It is our aim to remark that, owing to the structure of the Bethe lattice, the PM approach can be applied not only formally, but explicitly. This allows us to verify the exactness of type (i) and (ii) results, and also provides the complete analytical solution of a large class of Hamiltonian models, which include as very special cases the Ising model, the Potts model, the vector (or planar) Potts model, the Ashkin–Teller model, the  $Z(q)$  (or clock) model, and all the previous models with annealed site-dilution. In the following we describe briefly the essential properties of the PM approach; then we specialise it to Bethe lattices and prove that type (i) results do not converge to the correct limits. Finally, we give a summary of results concerning the solution of Hamiltonian models.

Let us consider a discrete system  $(V, Q)$  formed by the countable set  $V$  of points, and the finite set  $Q = \{1, 2, \dots, q\}$  of states that each point may assume. Each collection  $\{A_1, A_2, \dots, A_q\}$  of  $q$  finite subsets of  $V$  characterises one local event (with basis  $A \equiv \bigcup_{r \in Q} A_r$ ) on the system, i.e. the set of all the configurations of  $V$  such that every point  $i \in A_r$  is in the state  $r \in Q$ . The set of all local events on  $(V, Q)$  is a semi-ring  $R$  (for results of abstract measure theory we refer to the book by Zaanen (1967)), i.e. the simplest collection of sets where probability measures describing the system can be defined. Generalised local and non-local events also can be considered, since any probability measure  $\mu$  on  $R$  is extendible to the  $\sigma$ -field  $F \supset R$  of all  $\mu$ -measurable events. An interaction  $I$  in the system is a properly normed real function defined on the elements of  $R$ , which is zero on the global event. We suppose that  $V$  is a metric space and  $I(\mathcal{E}) = 0$  for every non-void  $\mathcal{E} \in R$  whose basis contains at least two points at distance larger than a fixed number  $m > 0$ . Given a finite subset  $V_0$  of  $V$ , we call the internal boundary  $\Delta V_0$  (external boundary  $\partial V_0$ ) of  $V_0$  the set of all the points in  $V_0$  ( $V - V_0$ ) such that their distance from at least one point in  $V - V_0$  ( $V_0$ ) is less than or equal to  $m$ . Given two local events  $\mathcal{E}_{V_0}$ ,  $\mathcal{E}_{\partial V_0}$ , respectively with basis  $V_0$  and  $\partial V_0$ , the Hamiltonian on  $V_0$  (relative to the 'configuration'  $\mathcal{E}_{V_0}$  of  $V_0$ ) is defined as

$$-\beta \mathcal{H}_{V_0}(\mathcal{E}_{V_0}) \equiv \sum_{\mathcal{E} \in R: \mathcal{E} \supseteq \mathcal{E}_{V_0}} I(\mathcal{E}) \quad (1)$$

while the interface Hamiltonian on  $\Delta V_0$  (relative to  $\mathcal{E}_{V_0}$  and the 'boundary condition'  $\mathcal{E}_{\partial V_0}$ ) is defined as

$$-\beta \mathcal{H}_{\Delta V_0}(\mathcal{E}_{V_0}; \mathcal{E}_{\partial V_0}) \equiv \sum_{\mathcal{E} \in R: \mathcal{E} \supseteq \mathcal{E}_{V_0} \cap \mathcal{E}_{\partial V_0}; \mathcal{E} \supseteq \mathcal{E}_{V_0}; \mathcal{E} \supseteq \mathcal{E}_{\partial V_0}} I(\mathcal{E}). \quad (2)$$

The (conditional) Gibbs probability measure of  $\mathcal{E}_{V_0}$  (given that  $\mathcal{E}_{\partial V_0}$  is an event with

probability 1) is obtained in the conventional way as

$$\mu_0(\mathcal{E}_{V_0} | \mathcal{E}_{\partial V_0}) \equiv \frac{\exp[-\beta \mathcal{H}_{V_0}(\mathcal{E}_{V_0}) - \beta \mathcal{H}_{\Delta V_0}(\mathcal{E}_{\partial V_0})]}{\sum_{\mathcal{E}_{V_0} \in R} \exp[-\beta \mathcal{H}_{V_0}(\mathcal{E}_{V_0}) - \beta \mathcal{H}_{\Delta V_0}(\mathcal{E}_{\partial V_0})]} \quad (3)$$

The knowledge of (3) for every  $\mathcal{E}_{V_0}$  (and fixed  $\mathcal{E}_{\partial V_0}$ ) is sufficient to extend  $\mu_0$  to the semi-ring  $R_0$  of all local events with basis equal to or included in  $V_0$ . Let  $\{V_n\}_{n=1}^{\infty}$  be a sequence of finite subsets of  $V$  (ordered by inclusion) such that  $\bigcup_{n=1}^{\infty} V_n = V$ ; let  $\mu_n$  be a probability measure on  $R_n$ ; and let  $l$  be an index such that  $A \subset V_n$  for every  $n > l$ . It has been proved (Ruelle 1978) that one can choose a subsequence  $\{V_{n'}\}_{n'=1}^{\infty}$  of  $\{V_n\}_{n=1}^{\infty}$  such that the limit

$$\lim_{l' < n' \rightarrow \infty} \mu_{n'}(\mathcal{E}_A) \equiv \mu(\mathcal{E}_A) \quad (4)$$

exists for every  $\mathcal{E}_A \in R$ , and defines the probability measure  $\mu$  on  $R$ , i.e. a thermodynamic limit of the  $\mu_n$ 's. If every  $\mu_n$  is a Gibbs probability measure (defined as previously described) on  $R_n$ , distinct limits may be obtained depending on the selected subsequence of boundary conditions (this would be the typical case at low temperatures, if two or more ordered phases arise in the system). We denote  $\mathcal{H}_l$  the closed convex hull of the set of all the limits defined through (3) and (4): the thermodynamic limits of the (non-conditional) probability measures obtained by deletion of the interface Hamiltonian in (3) also belong to  $\mathcal{H}_l$  (Ruelle 1978).

Remark that no special hypotheses were made about the properties of the sequences  $\{V_n\}_{n=1}^{\infty}$ . From the physical point of view this is justified by the intensive character of local events, whose distance from the boundaries of  $V_n$  becomes larger and larger in the thermodynamic limit. On the contrary, an extensive function such as the free energy receives contributions from  $\Delta V_n$  for every  $n$ , so its limit may depend on the choice of the sequence. Van Hove's convergence condition in the present terminology can be written as

$$\lim_{n \rightarrow \infty} |\Delta V_n| / |V_n| = 0 \quad (5)$$

(here  $|X|$  is the number of elements of the finite set  $X$ ). The sequences  $\{V_n\}_{n=1}^{\infty}$  which satisfy relation (5) give correct results for the entropy and free energy limits, while the others give results which do not agree with those of the PM approach. This is established (Ruelle 1978) for hypercubic lattices: we will show that the same is true for Bethe lattices.

Let  $L$  be a Bethe lattice with coordination number  $\sigma + 1$ , and let us make the following hypotheses: (a) the point set  $V$  is the set of sites of  $L = (V, E)$ ; (b) the distance between two sites is defined as the number of bonds in the walk connecting them; (c) the maximum interaction range is  $m = 1$ . Then it is easy to see that for every increasing sequence  $\{T_n\}_{n=1}^{\infty}$  of finite trees tending to  $L$  we have

$$\lim_{n \rightarrow \infty} |\Delta V_n| / |V_n| = (\sigma - 1) / \sigma \quad (6)$$

i.e. condition (5) does not hold. There is another trivial but important topological property of  $\{T_n\}_{n=1}^{\infty}$ . Let us denote by  $\psi_i(G)$  the local number of bonds per site relative to the site  $i$  of a locally finite graph  $G$ , i.e. half the coordination number of  $i$  in  $G$ . The mean number of bonds per site  $\psi(G)$  is defined as the average of  $\psi_i(G)$  on  $G$ .

Since  $\psi_i(L)$  is the same for every site of the Bethe lattice, we obtain

$$\psi(L) = \psi_i(L) = (\sigma + 1)/2 \tag{7}$$

while on  $T_n$  we find

$$\psi(T_n) = |V_n|^{-1} \sum_{i \in V_n} \psi_i(T_n) = 1 + 1/|V_n|. \tag{8}$$

Therefore the asymptotic mean topological properties of  $\{T_n\}_{n=1}^\infty$  do not recover those of  $L$ , since

$$\lim_{n \rightarrow \infty} \psi(T_n) = 1 \neq \psi(L). \tag{9}$$

Now, to see that relation (9) implies that type (i) methods do not agree with the PM approach, let us return to the consequences of hypotheses (a)–(c). They imply that the sum in (1) contains only terms of the type  $I(\mathcal{E}_{i\nu_i} \cap \mathcal{E}_{j\nu_j})$  and  $I(\mathcal{E}_{i\nu_i})$ , where  $\langle ij \rangle \in E$  and  $\mathcal{E}_{k\nu_k}$  is the local event which attributes the state  $\nu_k \in Q$  to the site  $k \in V$ . Therefore, the Hamiltonian (1) can be written (for every finite  $V_0 \subset V$ ) in the most familiar form:

$$-\beta \mathcal{H}_{V_0}(\mathcal{E}_{V_0}) = \sum_{\langle ij \rangle \in E_0} K_{(ij)\nu_i\nu_j} + \sum_{i \in V_0} H_{i\nu_i} \tag{10}$$

where  $E_0 \subset E$  is the set of all the bonds whose terminal sites belong to  $V_0$ ; the values of  $\nu_k$  are fixed by the condition  $\mathcal{E}_{k\nu_k} \supset \mathcal{E}_{V_0}$ ; and  $K$  and  $H$  are real functions of their arguments. Relation (10) implies that every Gibbs probability measure  $\mu_n$  on  $R_n$  is one-step Markov, as well as any thermodynamic limit  $\mu$  on  $R$ . It follows that every local event can be seen as a simple branching process (see e.g. Feller (1968) and Loève (1977)), so that its probability can be expressed in terms of the measures of local events having one site or two adjacent sites as basis ('site' and 'bond' probabilities):

$$\mu(\mathcal{E}_A) = \sum_{\mathcal{E}_A \in R} \mu(\mathcal{E}_{n\nu_n}) \prod_{|ij \rangle \in E_A} \mu(\mathcal{E}_{j\nu_j} | \mathcal{E}_{i\nu_i}). \tag{11}$$

Here  $T_A = (V_A, E_A) \subset L$  is the smallest tree containing  $A$ ;  $\bar{A} \equiv V_A - A$ ;  $h \in V_A$  is the source of the process, which propagates along the bonds  $|ij \rangle \in E_A$  oriented along the running direction; it is understood that  $\nu_k$  assumes the value fixed by  $\mathcal{E}_A$  ( $\mathcal{E}_{\bar{A}}$ ) for  $k \in A$  ( $k \in \bar{A}$ ); and, following the definition of conditional probabilities,  $\mu(\mathcal{E}_{j\nu_j} | \mathcal{E}_{i\nu_i}) = \mu(\mathcal{E}_{i\nu_i} \cap \mathcal{E}_{j\nu_j}) / \mu(\mathcal{E}_{i\nu_i})$ . Direct counting of the possible configurations of the system, and use of (11), give the entropy on  $T_n = (V_n, E_n)$  (Peruggi 1983b):

$$k^{-1} S_{T_n} = - \sum_{\langle ij \rangle \in E_n} \sum_{r,s=1}^q \mathcal{L}(\mu_n(\mathcal{E}_{ir} \cap \mathcal{E}_{js})) + \sum_{i \in V_n} \sigma_i \sum_{r=1}^q \mathcal{L}(\mu_n(\mathcal{E}_{ir})) \tag{12}$$

where  $k$  is the Boltzmann constant,  $\sigma_i + 1$  is the coordination number of the site  $i$  in  $T_n$ , and  $\mathcal{L}(a) \equiv a \ln a$ . The internal energy on  $T_n$  is easily obtained by (10):

$$\beta \mathcal{U}_{T_n} = - \sum_{\langle ij \rangle \in E_n} \sum_{r,s=1}^q K_{(ij)rs} \mu_n(\mathcal{E}_{ir} \cap \mathcal{E}_{js}) - \sum_{i \in V_n} \sum_{r=1}^q H_{ir} \mu_n(\mathcal{E}_{ir}). \tag{13}$$

The free energy on  $T_n$  is obviously given by

$$\beta \mathcal{F}_{T_n} = \beta \mathcal{U}_{T_n} - k^{-1} S_{T_n}. \tag{14}$$

Dividing (12) and (13) by  $|V_n| \equiv |E_n| / \psi(T_n)$ , and taking the thermodynamic limit, we obtain bond terms formed by the spatial average on  $E$  of  $\mathcal{L}(\mu(\mathcal{E}_{ir} \cap \mathcal{E}_{js}))$  and  $K_{(ij)rs} \mu(\mathcal{E}_{ir} \cap \mathcal{E}_{js})$ , respectively, times the factor  $\lim_{n \rightarrow \infty} \psi(T_n)$ , which does not converge

to the correct value  $\psi(L)$ . We deduce that the topological discrepancy (9) affects the thermodynamic limits of the entropy, internal energy, and free energy as an incorrect counting of bond contributions with respect to site contributions<sup>†</sup>. This explains the unusual physical properties of type (i) solutions, which could be attributed, at most, to a very big finite tree, but actually have no relation with Hamiltonian models on Bethe lattices.

Let us consider now the explicit solution of Hamiltonian models<sup>‡</sup>. We add the following hypothesis to (a)–(c): (d) the interaction  $I$  is rotationally and one-step translationally invariant on  $L$ §. Thus relations (10)–(14) still hold, but we also suppose that  $K_{\langle ij \rangle rs} = K_{rs}$  for every  $\langle ij \rangle \in E$ , and  $H_{ir} = H_r$  for every  $i \in V$ . One can prove that every Gibbs probability measure  $\mu_n$  on  $R_n$  is completely characterised by certain parameters  $\{\{\lambda_n(i; r)\}_{r=1}^q\}_{i \in V_n}$ . In fact all site and bond probabilities may be calculated by using these parameters, and univocally define  $\mu_n$  by means of (11). We remark that, by definition, all the  $\lambda$ 's are real positive, and  $\lambda_n(i; 1) = 1$  for every  $i \in V_n$ . The parameters  $\{\{\lambda_n(i; r)\}_{r=2}^q\}_{i \in \Delta V_n}$  are fixed by the boundary condition  $\mathcal{E}_{\partial V_n}$ ; all the others are obtained through hierarchical or recursive relations (depending on  $\mathcal{E}_{\partial V_n}$  and the structure of  $T_n$ ). The latter can be written in the form

$$\lambda'(r) = \exp(H_r - H_1) \left( \frac{\sum_{s=1}^q \exp(K_{rs}) \lambda(s)}{\sum_{s=1}^q \exp(K_{1s}) \lambda(s)} \right)^\sigma, \quad r = 2, \dots, q. \quad (15)$$

In the thermodynamic limit these iterative equations reach fixed points or two-step cyclic points, which characterise rotationally and one- or two-step translationally invariant probability measures on  $R$ , i.e. extremal points of  $\mathcal{X}_I$  which describe pure phases||. As regards the hierarchical relations, they give rise to the same probability measures, or to their mixtures ( $\equiv$  phase mixing), i.e. non-extremal points of  $\mathcal{X}_I$ . Let us consider the partition of  $V$  in two subsets  $V^e, V^o$  such that the sites adjacent to every  $i \in V^x$  belong to  $V^y$ ;  $x, y = e, o$  or  $o, e$ . At a two-step cyclic point characterised by the parameters  $\{\lambda^e(r), \lambda^o(r)\}_{r \in Q}$  we find

$$\mu(\mathcal{E}_{ir}) = \frac{\lambda^x(r) \sum_{t=1}^q \exp(K_{it}) \lambda^y(t)}{\sum_{s=1}^q \lambda^x(s) \sum_{t=1}^q \exp(K_{st}) \lambda^y(t)}, \quad \mu(\mathcal{E}_{js} | \mathcal{E}_{ir}) = \frac{\exp(K_{rs}) \lambda^y(s)}{\sum_{t=1}^q \exp(K_{rt}) \lambda^y(t)}, \quad (16)$$

where  $i \in V^x, \langle ij \rangle \in E, r \in Q, s \in Q$ , and  $x, y = e, o$  or  $o, e$ . At a fixed point we have  $\lambda^e(r) = \lambda^o(r)$  for every  $r \in Q$ , and sublattice dependence disappears. The (per site) free energy  $\beta\mathcal{F}$  on  $L$ , relative to pure phases, is found by taking the thermodynamic

<sup>†</sup> Note that this holds for any thermodynamic limit  $\mu$ . Then the suggestion by Peruggi *et al* (1983a), that type (i) results could correspond to probability measures which are not translationally invariant on  $L$ , is erroneous.

<sup>‡</sup> The proofs of all the following results will be found in Peruggi (1983a, b).

§ The most general case of a rotationally and two-step translationally invariant interaction is treated in full detail by Peruggi (1983a, b). Note that we do not refer to rotations and translations of a Euclidean space where a realisation of  $L$  is drawn, but to isomorphisms of the Bethe lattice onto itself, defined by means of lexicographic orderings of its sites.

|| As a matter of fact, see the solution of the Potts model by Peruggi *et al* (1983a). Also remark that, up to now, there is no general proof that relations (15) are piecewise contracting (PC), i.e. that every seed goes into a fixed or two-step cyclic point. If the PC property does not hold for a certain model, there could also be extremal points of  $\mathcal{X}_I$  related to pure states which are not invariant under rotations and/or translations of  $L$ . However: (A) the PC property holds for the ferromagnetic and antiferromagnetic Ising and Potts models; (B) Brouwer's theorem (see e.g. Collatz 1966) ensures that relations (15) always admit at least one fixed or two-step cyclic point; (C) numerical evidence strongly supports the PC property.

limit of (12) and (13), provided relations (16) and the heuristic rule  $\lim_{n \rightarrow \infty} \psi(T_n) \equiv (\sigma + 1)/2$  are used. To see that the result is correct, it is enough to perform simple checks: e.g. to verify that  $\partial(\beta\mathcal{F})/\partial H_r = \mu(\mathcal{E}_{ir})$ ,  $r \in Q$ , as was done by Peruggi *et al* (1983a). A rigorous limit procedure may be introduced, too, which gives the same expression for  $\beta\mathcal{F}$ .

Finally, let us remark that it is easy to see that the Bethe–Peierls approximation on standard lattices gives the same site and bond probabilities as (16) (thus justifying the name ‘Bethe lattices’), and that there is general agreement between our method and type (ii) results. In spite of this equivalence the PM approach has two relevant advantages. It gives directly the free energy, which has never been calculated with type (ii) methods except for one special case (Baxter 1982) where an integration procedure, starting from the equation of state, is used. Furthermore, using (11), one can calculate the probability measure of any event on the Bethe lattice, so that any problem on  $L$  can be solved, not only the study of thermodynamic properties. As a matter of fact, we have also found the complete analytic solution of polychromatic and  $\mathcal{AB}$  correlated-site/random-bond percolation problems (Peruggi 1983b, Peruggi *et al* 1983b).

To summarise, we have emphasised that the thermodynamic limit of probability measures in the canonical approach to the solution of Hamiltonian models on any discrete state countable lattice. This PM approach has allowed us to clarify the well known dichotomy concerning the solution of Hamiltonian models on Bethe lattices. The methods (i), which use the thermodynamic limit of the free energy, generate results not related to the systems under study. Other approaches (ii) give exact results, although they are criticisable from the formal point of view. We sketched the PM approach, and pointed out that, besides its formal coherence, it gives easily complete information about a large class of Hamiltonian models on Bethe lattices.

The author is very indebted to A Coniglio, B Preziosi, G Monroy and F di Liberto for many interesting and useful discussions.

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